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The Klein–Gordon equation in a Kerr–Newman background space

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Abstract. The time-dependent Klein–Gordon equation for a massive scalar meson field is examined in the Kerr–Newman background space and solutions near the outer horizon and at infinity are obtained in terms of Whittaker functions. In the special case of a Schwarzschild black hole the solutions are shown to be closely related to asymptotic solutions derived elsewhere.

1. Introduction

In a recent paper (Rowan and Stephenson 1976, to be referred to as II), solutions of the Klein–Gordon equation for a massive scalar meson field in the region exterior to a Schwarzschild black hole have been obtained by the Liouville–Green technique. This analysis required that the black hole be large (that is, that the ratio of the Schwarzschild radius to the Compton wavelength of the meson should be large) and that the energy satisfy a certain inequality relationship. In the present paper we obtain solutions of the radial Klein–Gordon equation for a massive meson field near the outer horizon and at infinity in the presence of a charged, rotating black hole of *arbitrary* mass as represented by the Kerr–Newman metric. The penalty we have to pay for allowing the mass to be arbitrary is that solutions can be obtained only in these regions rather than over the whole space as in II. The solutions are all expressible in terms of Whittaker functions. Finally the special case of a Schwarzschild black hole is examined in detail and the results are shown to be closely related to those obtained by other methods (see, for example, Gibbons 1975, DeWitt 1975, Boulware 1975 and Page 1976).

2. Basic equations

As in II we start with the Klein–Gordon equation

$$(\Box^2+\mu^2)\Phi=0, \qquad (2.1)$$

where, as usual, μ is the inverse Compton wavelength of the meson. In generally covariant form (2.1) is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{i\kappa}\frac{\partial\Phi}{\partial x^\kappa}\right)+\mu^2\Phi=0, \qquad (2.2)$$

where the metric $ds^2 = g_{i\kappa}\,dx^i\,dx^\kappa$ is assumed to have signature −2, and g is the determinant of $g_{i\kappa}$. Now a charged, rotating black hole of mass M, charge Q and angular velocity a is described by the Kerr–Newman metric (Misner *et al* 1973)

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta\,d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)\,d\phi - a\,dt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2\,d\theta^2, \tag{2.3}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta. \tag{2.4}$$

(These are the Boyer–Lindquist or generalized Schwarzschild coordinates—see Boyer and Lindquist 1967.)

Substituting (2.3) into (2.2) we obtain

$$\left[\frac{[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]}{\Delta}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial r}\left(\Delta\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right.$$
$$\left. - \frac{(\Delta - a^2\sin^2\theta)}{\Delta\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - 2a\frac{[\Delta-(r^2+a^2)]}{\Delta}\frac{\partial^2}{\partial\phi\,\partial t} + \rho^2\mu^2\right]\Phi = 0, \tag{2.5}$$

which, on writing

$$\Phi = R(r)S(\theta)\,e^{im\phi}\,e^{-i\omega t}, \tag{2.6}$$

reduces to

$$\frac{1}{R}\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + \frac{1}{S\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \frac{(\Delta - a^2\sin^2\theta)(-m^2)}{\Delta\sin^2\theta} + \frac{2a[\Delta-(r^2+a^2)]m\omega}{\Delta}$$
$$- \frac{[(r^2+a^2) - \Delta a^2\sin^2\theta]}{\Delta}(-\omega^2) - \rho^2\mu^2 = 0. \tag{2.7}$$

This in turn can be separated into two equations

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(\lambda_{lm} + c^2\cos^2\theta - \frac{m^2}{\sin^2\theta}\right)S = 0, \tag{2.8}$$

where $c^2 = a^2(\omega^2 - \mu^2)$, and

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + [\omega^2(r^2+a^2)^2 - 4Ma\omega mr + 2Q^2a\omega m - \mu^2r^2\Delta$$
$$+ m^2a^2 - (\omega^2a^2 + \lambda_{lm})\Delta]R = 0. \tag{2.9}$$

Equation (2.8) has as its solutions the oblate spheroidal harmonic functions $S_{lm}(ic, \cos\theta)$ with eigenvalue λ_{lm}, where l, m are integers such that $|m| \leq l$. (See, for example, Ford 1975 and Morse and Feshbach 1953.)

We now put

$$r_+ = M + [M^2 - (a^2+Q^2)]^{1/2}, \qquad r_- = M - [M^2 - (a^2+Q^2)]^{1/2}, \tag{2.10}$$

into the radial equation (2.9) to obtain

$$(r-r_+)(r-r_-)\frac{\mathrm{d}}{\mathrm{d}r}\left((r-r_+)(r-r_-)\frac{\mathrm{d}R}{\mathrm{d}r}\right)+[\omega^2(r^2+a^2)^2-4Ma\omega mr+2Q^2a\omega m$$
$$-\mu^2r^2(r-r_+)(r-r_-)+m^2a^2-(\omega^2a^2+\lambda_{lm})(r-r_+)(r-r_-)]R=0. \quad (2.11)$$

3. The radial equation

Defining x and d by

$$Mx=r-r_+, \qquad 2Md=r_+-r_-=2[M^2-(a^2+Q^2)]^{1/2}, \quad (3.1)$$

equation (2.11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(x+2d)\frac{\mathrm{d}R}{\mathrm{d}x}\right)+\left(\omega^2\frac{\{M^2[(x+d+1)^2-(d^2-1)]-Q^2\}^2}{M^2x(x+2d)}\right.$$
$$-\frac{4a\omega m(x+d+1)}{x(x+2d)}+\frac{2Q^2a\omega m}{M^2x(x+2d)}-M^2\mu^2(x+d+1)^2$$
$$\left.+\frac{m^2a^2}{M^2x(x+2d)}-(\omega^2a^2+\lambda_{lm})\right)R=0 \quad (3.2)$$

which, by writing $R(x)=Z(x)[x(x+2d)]^{-1/2}$, is put into the normal form

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2}+\left(M^2(\omega^2-\mu^2)+\frac{1}{M^2x^2(x+2d)^2}[\![\omega^2\{M^4[4(x+d+1)^2+4(x+d+1)x(x+2d)]\right.$$
$$-2M^2Q^2[x(x+2d)+2(x+d+1)]+Q^4\}-4a\omega mM^2(x+d+1)$$
$$+2Q^2a\omega m-\mu^2M^4[2x+(d+1)^2]x(x+2d)+m^2a^2$$
$$\left.-(\omega^2a^2+\lambda_{lm})M^2(x+2d)x+M^2d^2]\!]\right)Z=0. \quad (3.3)$$

No general solution in terms of standard functions is known for this equation over the range $0\leqslant x<\infty$. However, we may obtain solutions near $x=0$ and at infinity by the following method: we first write (3.3) in partial fraction form as

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2}+\left[M^2(\omega^2-\mu^2)+\frac{1}{M^2}\left(\frac{A}{x^2}+\frac{B}{x}+\frac{C}{(x+2d)^2}+\frac{D}{(x+2d)}\right)\right]Z=0, \quad (3.4)$$

where the constants A, B, C and D are found after considerable algebra to be given by:

$$A=\frac{1}{4d^2}[M^2d^2+m^2a^2+2Q^2a\omega m-4a\omega mM^2(d+1)+\omega^2Q^2$$
$$-4M^2\omega^2Q^2(d+1)+4M^4\omega^2(d+1)^2], \quad (3.5)$$

$$B=\frac{1}{4d^3}[4\omega^2M^4(d+1)^2(2d-1)-4\omega^2M^2Q^2(d^2-1)+4a\omega mM^2$$
$$-2\mu^2M^4(d+1)^2d^2-2d^2(\omega^2a^2+\lambda_{lm})M^2-M^2d^2-m^2a^2$$
$$-2Q^2a\omega m-\omega^2Q^4], \quad (3.6)$$

$$C=\frac{1}{4d^2}[4\omega^2M^4(d-1)^2+4M^2\omega^2Q^2(d-1)+4a\omega mM^2(d-1)+M^2d^2$$
$$+m^2a^2+2Q^2a\omega m+\omega^2Q^4], \tag{3.7}$$

and

$$D=\frac{1}{4d^3}[4\omega^2M^4(2d+1)(d-1)^2+4\omega^2M^2Q^2(d^2-1)-4a\omega mM^2$$
$$+2\mu^2M^4d^2(d-1)^2+2d^2(\omega^2a^2+\lambda_{lm})M^2+M^2d^2+m^2a^2$$
$$+2Q^2a\omega m+\omega^2Q^4]. \tag{3.8}$$

We now consider the forms of the solutions in the two limits $x\to 0$ and $x\to\infty$.

3.1. *Case 1.* $x\to 0$

By expanding the C and D terms in (3.4) we have for small x the equation

$$\frac{d^2Z}{dx^2}=\left[\left(M^2(\mu^2-\omega^2)-\frac{C}{4M^2d^2}-\frac{D}{2M^2d}\right)-\frac{1}{M^2}\left(\frac{A}{x^2}+\frac{B}{x}\right)+O(x)\right]Z. \tag{3.9}$$

We now define η as

$$\eta=2\sqrt{F}x, \tag{3.10}$$

where

$$F=\left(M^2(\mu^2-\omega^2)-\frac{C}{4M^2d^2}-\frac{D}{2M^2d}\right). \tag{3.11}$$

Neglecting terms of $O(x)$, (3.9) finally takes the form

$$\frac{d^2Z}{d\eta^2}=\left(\frac{1}{4}-\frac{B}{2M^2\sqrt{F}}\frac{1}{\eta}-\frac{A}{M^2\eta^2}\right)Z, \tag{3.12}$$

which is to be compared with the Whittaker equation

$$\frac{d^2Z}{d\eta^2}=\left(\frac{1}{4}-\frac{\kappa}{\eta}+\frac{\bar{m}^2-\frac{1}{4}}{\eta^2}\right)Z. \tag{3.13}$$

It follows that (3.12) has solutions in terms of the Whittaker functions with κ, $\bar{m}$ defined by

$$\kappa=B/2M^2\sqrt{F}, \qquad \bar{m}^2=\tfrac{1}{4}-(A/M^2). \tag{3.14}$$

3.2. *Case 2.* $x\to\infty$

We now expand the terms $(x+2d)^{-1}$ and $(x+2d)^{-2}$ in (3.4) for large x and write the equation in the form

$$\frac{d^2Z}{dx^2}+\left[M^2(\omega^2-\mu^2)+\frac{1}{M^2}\left(\frac{A+C-2dD}{x^2}+\frac{B+D}{x}\right)+O\left(\frac{1}{x^3}\right)\right]Z=0. \tag{3.15}$$

Defining a new independent variable ξ by

$$\xi = 2M(\mu^2 - \omega^2)^{1/2} x \tag{3.16}$$

and neglecting terms of $O(1/x^3)$, (3.15) becomes

$$\frac{d^2 Z}{d\xi^2} = \left(\frac{1}{4} - \frac{B+D}{2M^3(\mu^2-\omega^2)^{1/2}}\frac{1}{\xi} - \frac{A+C-2dD}{M^2\xi^2}\right)Z. \tag{3.17}$$

Comparing this with the Whittaker equation (3.13) we see that (3.17) has Whittaker solutions with κ, $\bar{m}$ defined by

$$\kappa = \frac{B+D}{2M^3(\mu^2-\omega^2)^{1/2}}, \qquad \bar{m}^2 = \frac{1}{4} - \frac{A+C-2dD}{M^2}. \tag{3.18}$$

In both cases 1 and 2 therefore, the asymptotic solutions of (3.4) near the origin and at infinity are expressed in terms of Whittaker functions.

4. Solutions in standard coordinates

4.1. Case 1

As $x \to 0$ we have $r \to r_+$ and the solutions of (2.11) are given by

$$R(r) \sim \begin{cases} \dfrac{M}{\Delta^{1/2}} M_{\kappa,\bar{m}}\left(\dfrac{2\sqrt{F}}{M}(r-r_+)\right), \\[2ex] \dfrac{M}{\Delta^{1/2}} M_{\kappa,-\bar{m}}\left(\dfrac{2\sqrt{F}}{M}(r-r_+)\right), \end{cases} \tag{4.1}$$

provided $F \neq 0$, where k, $\bar{m}$ are defined by (3.14) and $M_{\kappa,\pm\bar{m}}$ are Whittaker functions (see Whittaker and Watson 1927). These two functions form independent solutions of the Whittaker equation provided $2\bar{m}$ is not integral. In the case when $2\bar{m}$ is integral the second solution is taken as the Whittaker function $W_{\kappa,\bar{m}}$. There is no specific physical reason why $2\bar{m}$ with $\bar{m}$ defined by (3.14) should be integral and we shall therefore examine the solutions as $r \to r_+$ in terms of the $M_{\kappa,\pm\bar{m}}$ functions. For all z we have the series expansion

$$M_{\kappa,\bar{m}}(z) = z^{\bar{m}+\frac{1}{2}} e^{-\frac{1}{2}z}\left(1 + \frac{(\frac{1}{2}+\bar{m}-\kappa)}{1!(2\bar{m}+1)!}z + \frac{(\frac{1}{2}+\bar{m}-\kappa)(\frac{3}{2}+\bar{m}-\kappa)}{2!(2\bar{m}+1)(2\bar{m}+2)}z^2 + \dots\right) \tag{4.2}$$

so that as $r \to r_+$ we have the radial solutions

$$R(r) \sim \begin{cases} \dfrac{M}{\Delta^{1/2}} e^{-(\sqrt{F}/M)(r-r_+)}\left(\dfrac{2\sqrt{F}}{M}(r-r_+)\right)^{\frac{1}{2}+\bar{m}}, \\[2ex] \dfrac{M}{\Delta^{1/2}} e^{-(\sqrt{F}/M)(r-r_+)}\left(\dfrac{2\sqrt{F}}{M}(r-r_+)\right)^{\frac{1}{2}-\bar{m}}, \end{cases} \tag{4.3}$$

$(F \neq 0)$.

4.2. Case 2

As $x \to \infty$ so $r \to \infty$ and the solutions of (2.11) are given asymptotically by

$$R(r) \sim \begin{cases} \dfrac{M}{\Delta^{1/2}} W_{\kappa,\bar{m}}(2(\mu^2-\omega^2)^{1/2}(r-r_+)), \\ \dfrac{M}{\Delta^{1/2}} W_{-\kappa,\bar{m}}(-2(\mu^2-\omega^2)^{1/2}(r-r_+)), \end{cases} \tag{4.4}$$

where we have chosen to use the $W_{\kappa,\bar{m}}$ functions because of their simpler asymptotic behaviour as compared with the $M_{\kappa,\bar{m}}$ functions, $W_{\kappa,\bar{m}}(x)$ and $W_{-\kappa,\bar{m}}(-x)$ forming two independent solutions of the Whittaker equation. In (4.4) κ, $\bar{m}$ are defined by (3.18). Using the asymptotic form for large $|z|$ given by

$$W_{\kappa,\bar{m}}(z) \sim e^{-\frac{1}{2}z} z^{\kappa}\left(1+\sum_{n=1}^{\infty}\frac{[\bar{m}^2-(\kappa-\frac{1}{2})^2][\bar{m}^2-(\kappa-\frac{3}{2})^2]\dots[\bar{m}^2-(\kappa-\bar{m}-\frac{1}{2})^2]}{n!z^n}\right), \tag{4.5}$$

we finally have the radial solutions as $r \to \infty$ in the form

$$R(r) \sim \begin{cases} \dfrac{M}{\Delta^{1/2}} e^{-[(\mu^2-\omega^2)^{1/2}(r-r_+)]}[2(\mu^2-\omega^2)^{1/2}(r-r_+)]^{\kappa}, \\ \dfrac{M}{\Delta^{1/2}} e^{+[(\mu^2-\omega^2)^{1/2}(r-r_+)]}[-2(\mu^2-\omega^2)^{1/2}(r-r_+)]^{-\kappa}, \end{cases} \tag{4.6}$$

provided $\mu^2-\omega^2 \neq 0$.

5. Special cases

In the last section we have used the conventional forms of the series expansions and asymptotic forms for the Whittaker functions $M_{\kappa,\bar{m}}$ and $W_{\kappa,\bar{m}}$ to derive the forms of the radial solutions. However, as $F \to 0$ in (3.14) and $\mu^2 \to \omega^2$ in (3.18) so the appropriate values of κ become very large, or if κ is imaginary then $|\kappa|$ becomes very large. In these cases the series forms given in (4.2) and (4.5) are not very appropriate for numerical calculation and the asymptotic expansions appropriate to large κ, small x, and large κ, large x given by Slater (1960) and Olver (1974) should be used.

Failing cases exist also in the Whittaker function solutions when, for $x \to 0$ we have $F=0$, and for $x \to \infty$ when $\mu^2=\omega^2$. To deal with these two situations we have to return to the original equations of case 1 and case 2.

5.1. Case 1. $F=0$

Here (3.9) becomes

$$\frac{d^2Z}{dx^2}=\left[-\frac{1}{M^2}\left(\frac{A}{x^2}+\frac{B}{x}\right)+O(x)\right]Z. \tag{5.1}$$

Neglecting terms of $O(x)$ and comparing with the standard equation

$$\frac{d^2Z}{dx^2}=\left(\frac{\alpha^2-1}{4x^2}+\frac{\beta^2}{4x}\right)Z \tag{5.2}$$

which has solutions

$$Z=\begin{cases} x^{1/2}I_\alpha(\beta x^{1/2}), \\ x^{1/2}K_\alpha(\beta x^{1/2}), \end{cases} \tag{5.3}$$

where I_α and K_α are modified Bessel functions of order α of the first and second kind respectively, we see that

$$\alpha^2 = 1-(4A/M^2), \qquad \beta^2=-4B/M^2. \tag{5.4}$$

The final radial solutions of (2.11) in this case are therefore

$$R(r)\sim\begin{cases} \dfrac{M^{1/2}}{\Delta^{1/2}}(r-r_+)^{1/2}I_\alpha\left(\dfrac{\beta}{M^{1/2}}(r-r_+)^{1/2}\right), \\ \dfrac{M^{1/2}}{\Delta^{1/2}}(r-r_+)^{1/2}K_\alpha\left(\dfrac{\beta}{M^{1/2}}(r-r_+)^{1/2}\right). \end{cases} \tag{5.5}$$

5.2. Case 2. $\mu^2=\omega^2$

Equation (3.15) now becomes

$$\frac{d^2Z}{dx^2}+\left[\frac{1}{M^2}\left(\frac{A+C-2dD}{x^2}+\frac{B+D}{x}\right)+O\left(\frac{1}{x^3}\right)\right]Z=0. \tag{5.6}$$

Neglecting terms of order $1/x^3$ we have

$$\frac{d^2Z}{dx^2}=-\left(\frac{A+C-2dD}{M^2x^2}+\frac{B+D}{M^2x}\right)Z \tag{5.7}$$

and comparing with (5.2) we have

$$\alpha^2=1-\frac{4}{M^2}(A+C-2dD), \qquad \beta^2=-\frac{4}{M^2}(B+D). \tag{5.8}$$

The radial solutions of (2.11) are therefore

$$R(r)\sim\begin{cases} \dfrac{M^{1/2}}{\Delta^{1/2}}(r-r_+)^{1/2}I_\alpha\left(\dfrac{\beta}{M^{1/2}}(r-r_+)^{1/2}\right), \\ \dfrac{M^{1/2}}{\Delta^{1/2}}(r-r_+)^{1/2}K_\alpha\left(\dfrac{\beta}{M^{1/2}}(r-r_+)^{1/2}\right). \end{cases} \tag{5.9}$$

6. The Schwarzschild black hole

We now examine the special case of $Q=0$, $a=0$ so that the metric (2.3) reduces to the Schwarzschild form. Accordingly we have

$$\Delta=r(r-2M), \qquad \rho=r, \qquad r_+=2M, \qquad r_-=0, \qquad d=1. \tag{6.1}$$

Then from (3.5)–(3.8)

$$A=\tfrac{1}{4}M^2+4M^4\omega^2, \tag{6.2}$$

$$B=4\omega^2M^4-2\mu^2M^4-\tfrac{1}{4}M^2-\tfrac{1}{2}M^2l(l+1), \tag{6.3}$$

$$C=\tfrac{1}{4}M^2, \tag{6.4}$$

$$D=\tfrac{1}{4}M^2+\tfrac{1}{2}M^2 l(l+1). \tag{6.5}$$

Hence for case 1, $x\to 0$, $F\neq 0$, we have

$$\kappa=-\frac{(2M^2\mu^2-4M^2\omega^2-\frac{1}{2}l(l+1)-\frac{1}{4})}{2[M^2(\mu^2-\omega^2)-\frac{1}{4}l(l+1)-\frac{3}{16}]^{1/2}}, \qquad \bar{m}=\pm 2\mathrm{i}M\omega, \tag{6.6}$$

where

$$F=[M^2(\mu^2-\omega^2)-\tfrac{1}{4}l(l+1)-\tfrac{3}{16}]^{1/2}. \tag{6.7}$$

Likewise for case 2, $x\to\infty$, $\mu\neq\omega$,

$$\kappa=M(2\omega^2-\mu^2)/(\mu^2-\omega^2)^{1/2}, \qquad \bar{m}^2=\tfrac{1}{4}-4M^2\omega^2+l(l+1). \tag{6.8}$$

Consider now, as an example, the radial solutions as $r\to\infty$ as given by (4.6) and (6.8). These equations lead to

$$R(r)\sim\frac{M}{[r(r-2M)]^{1/2}}\,\mathrm{e}^{\mp[(\mu^2-\omega^2)^{1/2}(r-2M)]}[\pm 2(\mu^2-\omega^2)^{1/2}(r-2M)]^{\pm M(2\omega^2-\mu^2)/(\mu^2-\omega^2)^{1/2}} \tag{6.9}$$

which can be written as

$$R(r)\sim\frac{A}{r}\exp\left\{\pm\mathrm{i}\left[\ln\left(1-\frac{2M}{r}\right)\right]\frac{M(2\omega^2-\mu^2)}{(\omega^2-\mu^2)^{1/2}}\right\}\left(1-\frac{2M}{r}\right)^{-1/2}$$
$$\times\exp\left[\pm\mathrm{i}\left((\omega^2-\mu^2)^{1/2}r+\frac{M(2\omega^2-\mu^2)}{(\omega^2-\mu^2)^{1/2}}\ln 2(\omega^2-\mu^2)^{1/2}r\right)\right] \tag{6.10}$$

where A is a constant.

Now the asymptotic form of the radial solutions given by Gibbons (1975) is (putting $Q=0$ in his work):

$$R(r)\sim\frac{1}{r}\exp\left[\pm\mathrm{i}\left((\omega^2-\mu^2)^{1/2}r+\frac{M(2\omega^2-\mu^2)}{(\omega^2-\mu^2)^{1/2}}\ln[2(\omega^2-\mu^2)^{1/2}r]\right)\right]. \tag{6.11}$$

Consequently we see that (6.10) differs from (6.11) by the multiplying factor

$$\exp\left\{\pm\mathrm{i}\left[\ln\left(1-\frac{2M}{r}\right)\right]\frac{M(2\omega^2-\mu^2)}{(\omega^2-\mu^2)^{1/2}}\right\}\left(1-\frac{2M}{r}\right)^{-1/2}. \tag{6.12}$$

For $\omega^2<\mu^2$ (6.12) reduces to a real power of $[1-(2M/r)]$, whilst for $\omega^2>\mu^2$ it retains the complex exponential form. In both cases as $r\to\infty$ (6.12) tends to unity, and (6.11) and (6.10) therefore have the same limit.

7. Conclusion

The work of the present paper has illustrated the difficulty of solving the radial equation for a massive scalar meson field in the background space of the most general black hole. In the absence of exact solutions we have found that unless the black hole is assumed to be large (as in II, in which case the Liouville–Green asymptotic method can be used) solutions can be obtained only at the ends of the range $0\leqslant x<\infty$. Even the much more special case of a massless meson field in a Schwarzschild space presents insurmountable difficulties (see, for example, Persides 1974) and no exact solution is known over the whole range. The theory developed here leads to a neat representation of the solutions at the ends of the range in terms of standard functions in the most general case, and for many black hole calculations such solutions are all that are required.

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